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The second result is that, for a broader class of probability of voting functions, the convergent equilibrium exists if and only if a strict symmetry condition on the candidate utility functions is satisfied. However, I show that both probability of win-maximizers and voteshare-maximizers satisfy this condition. Hence, policy coincidence will not exist in the general case, although Banks and Duggan (2005) have shown that every pure-strategy Nash equilibrium will exhibit convergence for voteshare-maximizing candidates and a broad class of probability of voting functions.

Policy convergence in a two-candidate probabilistic voting model.

Alexei V. Zakharov*

August 10, 2008

Abstract

I propose a generalization of the probabilistic voting model in two-candidate elections. Unlike in all previous works, I assume that the candidates have general von Neumann-Morgenstern utility functions defined over the voting outcomes. For the three-voter case, I derive necessary and sufficient conditions for the existence of a local Nash equilibrium in which the policy platforms of the candidates are identical. The first result is that, for a narrow class of probability of voting functions, the convergent equilibrium exists for all candidate utility functions, and corresponds to the mean-voter equilibrium of the earlier works. The second result is that, for a broader class of probability of voting functions, the convergent equilibrium exists if and only if a strict symmetry condition on the candidate utility functions is satisfied. However, I show that both probability of win-maximizers and voteshare-maximizers satisfy this condition. Hence, policy coincidence will not exist in the general case, although Banks and Duggan (2005) have shown that every pure-strategy Nash equilibrium will exhibit convergence for voteshare-maximizing candidates and a broad class of probability of voting functions.

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1 Introduction

A well-known result in the spatial voting theory is the mean voter theorem, due to Hinich, Ledyard, and Ordeshook (1972) and Hinich (1977,1978). In a nutshell, the theorem's statement is as follows. Suppose the probability of a voter supporting a party (or candidate) depends on the difference between the utilities that he attributes to the two parties. Suppose that the marginal effect of an increase in the utility difference on the probability of voting is equal across voters. Then, if the candidates maximize expected voteshare, the first-order conditions for Nash equilibrium are met if the two candidates select identical policy positions. Moreover, that policy position should maximize the sum of voter utilities.

The early works considered the voters to have quadratic disutility from policy distance. Under such assumptions, the convergent equilibrium is located at the mean of the voter ideal policies. Recent generalizations include generalized form of Eucledian preferences in Lin, Enelow, and Dorussen (1999), and strategic voting behavior in McKelvey and Patty (2006). In all works a convergent equilibrium was shown to exist in the general case[1]. Schofield (2007) looked at probabilistic voting with quadratic disutility when the candidates have different level of nonpolicy characteristics (valence), which is shown to affect the stability of the convergent equilibrium, but not its location or the fact that the first-order conditions are satisfied at the mean of the voter ideal policies.

Most works utilizing probabilistic voting models assume that the candidates are expected voteshare maximizers, or probability of victory maximization. The equivalence of candidate behavior under these two assumptions attracted attention of several scholars. Hinich (1977), Ledyard (1984) and Duggan (2000) argued in favor of the strategic equivalence of these two assumptions under Eucledian voter preferences and additive uncertainty. However, Patty (2005, 2007) demonstrated that under more general assumptions about the probability of voting functions, the response functions of probability-of-victory maximizers are different from those of expected voteshare maximizers, unless some very special

[1]Recent work by Patty, Snyder, and Ting (2008) demonstrate policy divergence in a multi-candidate probabilistic setting with strategic voters.

conditions on the voting probabilities are met.

The paper to which I compare most of my results is Banks and Duggan (2005), where the candidates are restricted to be voteshare maximizers, but there is a continuum of voters. In my work, I allow for general candidate utility functions, but there are only three voters.

In this work, I derive two main results. First, it is the necessary and sufficient conditions on the probability of vote functions that guarantee the existence of a convergent equilibrium under any candidate utility functions. I show that three conditions must be satisfied, as long as the candidate policy positions are identical. Firstly, the voters must be equally biased toward one of the candidates. Secondly, for each voter, a small change in the position of Candidate 1 must have the same effect as an equal, but opposite change in the position of Candidate 2. Finally, the preferences must be satiated. Roughly, that means that the probability of a voter supporting a candidate must be small if the candidate's policy platform is too extreme. These conditions are satisfied, for example, by the probability of vote functions that are induced by a utility function with an additive bias that is equal across the voters (such as in Hinich (1987) and later works).

Second, it is the necessary and sufficient conditions on the candidates utility functions that guarantee the existence of a convergent equilibrium for any set of voter response functions. These conditions are shown to be very restrictive. In particular, if we assume that the preferences of the two candidates are identical, then both candidates must be indifferent between fair lotteries offering 0 or 3 votes, or 1 and 2 votes, respectively. However, most types of political agents considered in the previous literature — like the expected utility maximizers, the probability of victory maximizers, and the plurality maximizers — are special cases that satisfy this exact condition.

2 The model and the results

There are 2 candidates and 3 stochastic voters. The candidates engage in a one-shot game by choosing policy platforms y_j from a compact policy space $X \subset \mathbf{R}^k$, $j = 1, 2$. The voters have policy preferences manifested in the twice continuously differentiable probability of voting functions $p_i : X^2 \to (0,1)$, where $p_i(y_1, y_2)$ is the probability that voter i will support candidate 1 given policy platforms y_1 and y_2. I assume that the votes are independent.

The payoff of a candidate depends on the number of votes she receives. Each candidate is endowed with a von Neumann-Morgenstern utility function. The expected utilities of the candidates are

$$U_1 = \sum_{l=0}^{3} P_l u_l^1, \quad (1)$$

$$U_2 = \sum_{l=0}^{3} P_l u_{3-l}^2, \quad (2)$$

where P_l is the probability that candidate 1 obtains exactly l votes, and u_l^j is the corresponding payoff to candidate j.

Simplifying the notation, we have

$$P_0 = (1-p_1)(1-p_2)(1-p_3), \quad (3)$$

$$P_1 = p_1(1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) + (1-p_1)(1-p_2)p_3, \quad (4)$$

$$P_2 = p_1p_2(1-p_3) + p_1(1-p_2)p_3 + (1-p_1)p_2p_3, \quad (5)$$

and

$$P_3 = p_1p_2p_3. \quad (6)$$

If y_1 maximizes $\sum_{l=0}^{3} P_l u_l^1$, then it also maximizes $\sum_{l=1}^{3} p_l(bu_l^1 + a) = b\sum_{l=1}^{3} p_l u_l^1 + a$ for any a and $b > 0$. Hence we can take $u_0^j = 0$ and $u_3^j = 1$, $j = 1, 2$, without loss of generality.

It is straightforward to show that an expected voteshare maximizer with $U_1 = p_1 + p_2 + p_3$ will have $u_1^1 = \frac{1}{3}$ and $u_2^1 = \frac{2}{3}$. A probability of victory maximizer will have $u_0^1 = u_1^1 = 0$ and $u_2^1 = u_3^1 = 1$. An expected plurality maximizer will also have $u_1^1 = \frac{1}{3}$ and $u_2^1 = \frac{2}{3}$.

I now define the solution concept that I am going to use.

Definition. (y_1^*, y_2^*) is a local Nash equilibrium in the 2-player game with the strategy set X and payoffs (1), (2) if there exists $\epsilon > 0$ such that for every $|y_1' - y_1^*| < \epsilon$, for every

$|y_2' - y_2^*| < \epsilon$, we have $U_1(y_1^*, y_2^*) \geq U_1(y_1', y_2^*)$ and $U_2(y_1^*, y_2^*) \geq U_2(y_1^*, y_2')$. The equilibrium is *interior* if y_1^*, y_2^* lie in the interior of X. It is *convergent* if $y_1^* = y_2^*$. (y_1^*, y_2^*) is a *critical* equilibrium if the first-order conditions

$$D_j(U_j) = 0 \tag{7}$$

are satisfied for $j = 1, 2$, where D_j denoted the gradient with respect to y_j.

The notion of critical equilibrium appears, for example, in Schofield and Sened (2006). If the candidate utilities are differentiable, a critical equilibrium is a weaker notion than a local Nash equilibrium or a global Nash equilibrium.

The first-order conditions for a critical equilibrium are

$$D_1(U_1) = u_1^1 D_1(P_1) + u_2^1 D_1(P_2) + D_1(P_3) = 0 \tag{8}$$

$$D_2(U_2) = u_1^2 D_2(P_2) + u_2^2 D_2(P_1) + D_2(P_0) = 0. \tag{9}$$

where D_i denotes the partial derivative with respect to candidate i's policy platform.

I will operate with the following assumptions about probability of vote functions.

Definition. The voters are *neutral* if for all y_1, y_2, we have $p_i(y_1, y_2) = 1 - p_i(y_2, y_1)$. The voters are *equally biased at convergent positions* if for all $y_1 = y_2$ we have $p_1(y_1, y_2) = p_2(y_1, y_2) = p_3(y_1, y_2)$. The voter is *marginally neutral at convergent positions* if $D_1(p_i(y_1, y_2)) = -D_2(p_i(y_1, y_2))$ for all $y_1 = y_2$. The voter i is *satiated* if for all $z \in X$ there exists $\bar{\alpha} > 0$, $\bar{\beta} < 0$ such that for every $r \in \mathbf{R}^k$, $\|r\| = 1$, we have $z + \bar{\alpha} r \in X$ and $r \cdot D_1(p_i(z + \alpha r, z + \alpha r)) < \bar{\beta}$ for all $\alpha > \bar{\alpha}$.

The satiating condition implies that if both candidates choose some common policy that is too distant from some fixed policy position, then the gradient of the probability of voting function with respect to either candidate's position will point toward that policy position. This property is illustrated on Figure 1.

I will now relate the above definitions to some well-known conditions on the probability of voting functions that appear in Banks and Duggan (2005). The first example is the utility difference model, when the assigns a certain utility to each policy platform, and

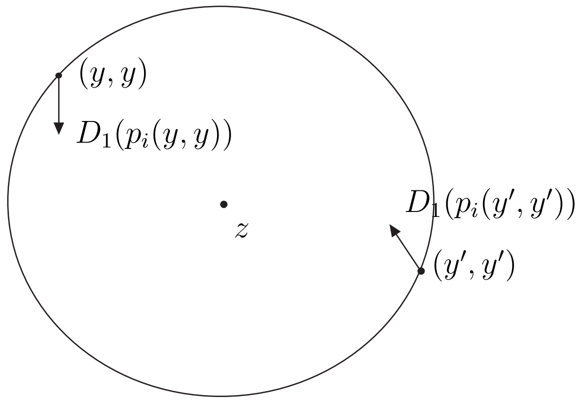

Figure 1: The preferences of voter i are satiated.

the probability of voting for candidate 1 depends on the difference between the utilities from the platforms of candidates 1 and 2.

Example 1 *Suppose that p_i satisfy the* utility difference model, *namely,*

$$p_i(y_1, y_2) = \hat{p}_i(u_i(y_1) - u_i(y_2)), \tag{10}$$

where $\hat{p}_i(\cdot)$ is a non-decreasing differentiable function, and $u_i(\cdot)$ is a differentiable function.[2]

Then, i is neutral if and only if $\hat{p}_i(x) = 1 - \hat{p}_i(-x)$ for all x; voters are equally biased at convergent positions if $\hat{p}_i(0) = \hat{p}^0$ for all i; all voters are marginally neutral at convergent positions; voter i is satiated if and only if there exist $v_i \in X$, $\bar{\alpha}_i > 0$, and $\bar{\beta}_i < 0$ such that for all $r \in \mathbf{R}^k$, $\|r\| = 1$, we have $z + \bar{\alpha}_i r \in X$ and $r \cdot D(u_i(v_i + \alpha r)) < \bar{\beta}_i$ for all $\alpha > \bar{\alpha}_i$.

Hence for the utility difference model the satiability of $p_i(\cdot)$ is equivalent to the satiability of $u_i(\cdot)$. The second example is a special case of the utility difference model.

Example 2 *Suppose that p_i satisfy the conditions of the* probabilistic voting model with generalized Eucledian preferences, *namely, voter i supports candidate 1 if*

$$-(y_1 - v_i)A_i(y_1 - v_i)^T + \delta_i + \epsilon_i < -(y_2 - v_i)A_i(y_2 - v_i)^T, \tag{11}$$

[2]This definition is slightly different from Banks and Duggan (2005), as I assume no possibility of abstention for the voters.

where $v_i \in X$ is the ideal policy of voter i, A_i is a non-negative definite $n \times n$ matrix, ϵ_i is a zero-mean random variable, and δ_i is the bias of voter i in favor of candidate 1.

Then, i is neutral if ϵ_i if and only if it is symmetric and $\delta_i = 0$; voters are equally biased at convergent positions if and only if $\delta_i = 0$ for all i; all voters are marginally neutral at convergent positions and all voters are satiated.

The final example is when the probability of voting for candidate 1 depends on the ratio of the utilities from the platforms of candidates 1 and 2. This specification first appeared in Coughlin and Nitzan (1981).

Example 3 *Suppose that p_i satisfy the* utility ratio model*, namely,*

$$p_i(y_1, y_2) = \hat{p}_i(u_i(y_1)/u_i(y_2)), \tag{12}$$

where $\hat{p}_i(\cdot)$ is a non-decreasing differentiable function, and $u_i(\cdot) > 0$ is a differentiable function.

Then, i is neutral if and only if $\hat{p}_i(x) = 1 - \hat{p}_i(\frac{1}{x})$ for all $x \neq 0$; voters are equally biased at convergent positions if $\hat{p}_i(1) = \hat{p}^1$ for all i; all voters are marginally neutral at convergent positions; voter i is satiated if and only if there exist $v_i \in X$, $\bar{\alpha}_i > 0$, and $\bar{\beta}_i < 0$ such that $r \in \mathbf{R}^k$, $\|r\| = 1$, $r \cdot D(u_i(v_i + \alpha r)) < \bar{\beta}_i$ for all $\alpha > \bar{\alpha}_i$.

The first result of this paper is that voter neutrality and satiated preferences are sufficient to guarantee the existence of a convergent equilibrium for all utility functions.

Proposition 1 *Suppose that for all i, the voters are equally biased and marginally neutral at convergent positions. Then a local, interior, convergent Nash equilibrium exists at any $y_1 = y_2 = y^*$ such that*

$$D_1(p_1) + D_1(p_2) + D_1(p_3) = 0. \tag{13}$$

If the preferences of the voters are satiated, then such an equilibrium exists.

This is a generalization of the "mean-voter" theorem to account for more general preferences of the candidates. Note that the location of the equilibrium does not depend on the values of the u_l^js. As a special case, I redefine the results in the more familiar terms of voter utilities.

Corollary 1 *Suppose that in the utility difference model,* $\hat{p}_i(\cdot)$ *are strictly increasing, and are equally biased at convergent positions. Suppose, moreover, that* $\hat{p}_i(0)$ *are equal for all* i*. Let* $u_i(\cdot)$ *be satiated. Then, any local, interior, convergent Nash equilibrium* y^* *is a local maximum of the total utility of voters*

$$U_U = u_1(y) + u_2(y) + u_3(y). \tag{14}$$

Moreover, suppose that $u_i(\cdot)$ *are concave. Then, the unique* y^* *is also the unique maximum of* U*.*

This result is similar to Theorem 6 and Corollary 3 of Banks and Duggan (2005). I show (only for three voters, though) that for a certain class of probability of voting functions, a convergent local Nash equilibrium exists and corresponds to the maximum of the sum of voter utility functions. Unlike in the previous works, I demonstrate that this result holds for all possible von Neumann-Morgenstern candidate utility functions defined over the voting outcomes.[3]

A result similar to the mean voter theorem was demonstrated by Coughlin and Nitzan (1981) for the utility ratio model. They have shown that voteshare-maximizing candidates in a pure-strategy Nash equilibrium both choose the same policy that maximizes the product of voter utilities — that is, the Nash bargaining solution. Here I show that this result carries over to the three-voter models with the candidates with general utility functions.

Corollary 2 *Suppose that in the utility ratio model,* $\hat{p}_i(\cdot)$ *are strictly increasing, and are equally biased at convergent positions. Suppose, moreover, that* $\hat{p}_i(1)$ *are equal for all* i*. Let* $u_i(\cdot)$ *be satiated. Then, any local, interior, convergent Nash equilibrium* y^* *is a local maximum of*

$$U_N = u_1(y)u_2(y)u_3(y). \tag{15}$$

Moreover, suppose that $u_i(\cdot)$ *are concave. Then, the unique* y^* *is also the unique maximum of* U*.*

[3] See also Patty (2005), where he derives the conditions on the probability of vote functions for which voteshare maximization and probability of win maximization yield identical convergent equilibria.

I now want to find the necessary and sufficient conditions on u_l^js for the existence of a local Nash equilibrium if the voters are not neutral and not equally biased at convergent positions. I still assume marginal neutrality at convergent positions.

Proposition 2 *Suppose that the voters are marginally neutral and the candidates have*

$$u_1^1 = 1 - u_2^2 \text{ and } u_2^1 = 1 - u_1^2. \tag{16}$$

The necessary and sufficient conditions for a convergent local equilibrium to exist at some y^* is that for some $z \in X$, there exists $\alpha > \|z - y^*\|$ such that for all $r \in \mathbf{R^k}$, $\|r\| = 1$, we have $z + \alpha r \in X$ and

$$\begin{aligned} & r \cdot D_1(U_1) = \\ = \quad & r \cdot D_1(p_1)(u_1^1(1 - 2p_2 - 2p_3 + 3p_2p_3) + u_2^1(p_2 + p_3 - 3p_2p_3) + p_2p_3) + \\ + \quad & r \cdot D_1(p_2)(u_1^1(1 - 2p_1 - 2p_3 + 3p_1p_3) + u_2^1(p_1 + p_3 - 3p_1p_3) + p_1p_3) + \\ + \quad & r \cdot D_1(p_3)(u_1^1(1 - 2p_1 - 2p_2 + 3p_1p_2) + u_2^1(p_1 + p_2 - 3p_1p_2) + p_1p_2) < 0 \end{aligned} \tag{17}$$

for $y_1 = y_2 = z + \alpha r$. If the conditions (16) are not satisfied, then a convergent local equilibrium does not exist for almost all probability of voting functions.

A convergent equilibrium in a k-dimensional policy space requires the solution of $2k$ first-order conditions (13), (9) in k unknowns, and will not exist in general. If the condition (16) is satisfied, or the voters are equally biased, the k equations (9) become redundant.

The conditions (16) are severe. On average, we should not expect these conditions to be satisfied (see Figure 2).

These conditions have an interpretation. Suppose that $L(p)$ is some lottery that offers 1 votes with probability $1 - p$ and 2 votes with probability P, and $M(p')$ is a lottery that offers 0 votes with probability $1 - p'$ and 3 votes with probability p'. If (16) hold, then there exists x such that candidate 1 prefers L to $M(p')$ if and only if candidate 2 prefers L to $M(p' + x)$ for all p, p'.

If the utility functions of the two candidates are identical, then they must also be symmetric:

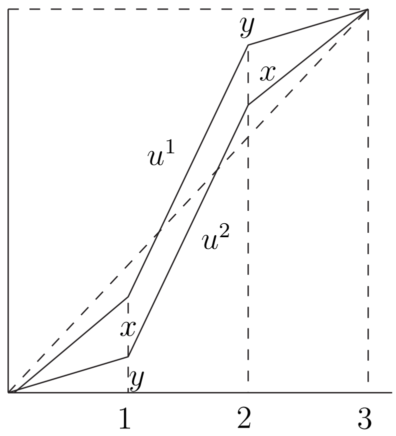

Figure 2: An example of u^1, u^2 satisfying conditions (16).

Corollary 3 *Suppose that* $u^1_l = u^2_l$ *for all* $l = 0, 1, 2, 3$ *and the voter preferences are satiated. Then a convergent equilibrium exists if and only if* $u^1_1 = 1 - u^1_2$.

Stated verbally, in order for the convergent equilibrium to exist given that the candidates have identical utility functions, both candidates must be indifferent between the fair lottery that offers 0 or 3 votes, and the fair lottery that offers 1 or 2 votes. If the conditions (16) are violated, then the convergent equilibrium does not exist for almost all probability of voting functions, since both conditions (13) and (9) must be satisfied at the equilibrium, rendering a system of $2k$ equations in k unknowns.

Note that conditions (16) are satisfied if both candidates are voteshare maximizers, or if both candidates are victory probability maximizers. It follows that both common assumptions made about the objective functions of the candidates in two-candidate models in fact satisfied a knife-edge condition. A convergent equilibrium will almost certainly not exist under all other objective functions. For example, (16) is violated if one candidate is a voteshare maximized and the other is a victory probability maximizer.

If (16) are satisfied, then the existence of a convergent equilibrium depends on the satiating condition (17) on the utility of a candidate. As we can see from the previous corollary, the condition is rather loose: for voters with satiated preferences, it is always satisfied if the candidates have identical objective functions. If the objective functions are not identical, one can derive several convenient sufficient conditions that guarantee a convergent local equilibrium.

Corollary 4 *Suppose (16) hold. Then, a convergent equilibrium exists if the preferences are satiated and either of the two conditions is satisfied:*

1. $u_1^1 < \frac{1}{3}$ *and* $u_2^1 \in [2u_1^1, u_1^1 + \frac{1}{3}]$, *or*

2. *For any* $i_1, i_2 \in \{1, 2, 3\}$, *we have* $p_{i_1} \leq \frac{1}{2}$ *and* $p_{i2} \leq \frac{1-2p_{i_1}}{2-3p_{i_1}}$.

These conditions are far less restrictive, and we can expect a convergent equilibrium to exist if the candidates have different utility functions.

3 Discussion

One should compare Proposition 2 with Theorem 6 of Banks and Duggan (2005), where the authors show that for voteshare maximizing candidates and utility difference, any pure-strategy Nash equilibrium must be convergent, as long as certain concavity conditions are satisfied. I show that a local pure-strategy Nash equilibrium exists for a narrow class of candidate utility functions under even some more general conditions on the probability of vote functions. The proof of my result is also non-constructive. However, for the general case of candidate utility functions, an additional condition is required — the voters must be equally biased toward the candidates if the policy platforms of the candidates coincide.

On one hand, my results demonstrate that an equilibrium where the candidates choose identical policy positions exists under a broader range of conditions than previously thought. For example, my work shows that it is reasonable to expect a convergent equilibrium if the biases of the voters toward one of the candidates are different, as long as the candidates are voteshare maximizers or plurality maximizers, like in some previous works, such as Banks and Duggan (2005).

On the other hand, if the candidate objective functions change, the convergent equilibrium unravels. Thus my work suggests an additional source of policy divergence in spatial voting models: objective functions of the candidates. In previous works, Schofield and Sened (2006) focused on the existence of political activists, Groseclose (2001) assumed partial policy motivation, and Zakharov (2008) looked at costly endogenous valence and strategic behavior by the candidates. The results of this work suggest that the cause of policy divergence may be more basic.

Appendix. Proofs of statements.

Proof of Proposition 1.

We have

$$D_1(P_0) = D_1(p_1)(p_2+p_3-p_2p_3-1)+D_1(p_2)(p_1+p_3-p_1p_3-1)+ \\ + D_1(p_3)(p_1+p_2-p_1p_2-1), \tag{18}$$

$$D_1(P_1) = D_1(p_1)(1-2p_2-2p_3+3p_2p_3)+D_1(p_2)(1-2p_1-2p_3+ \\ + 3p_1p_3)+D_1(p_3)(1-2p_1-2p_2+3p_1p_2), \tag{19}$$

$$D_1(P_2) = D_1(p_1)(p_2+p_3-3p_2p_3)+D_1(p_2)(p_1+p_3- \\ - 3p_1p_3)+D_1(p_3)(p_1+p_2-3p_1p_2), \tag{20}$$

and

$$D_1(P_3) = D_1(p_1)p_2p_3+D_1(p_2)p_1p_3+D_1(p_3)p_1p_2. \tag{21}$$

Put

$$p = p_1(y,y) = p_2(y,y) = p_3(y,y) \tag{22}$$

and

$$G = D_1(p_1)+D_1(p_2)+D_1(p_3). \tag{23}$$

We have

$$D_1(P_0) = G(2p-p^2-1),\ D_1(P_1) = G(1-4p+3p^2), \tag{24}$$

$$D_1(P_2) = G(2p-3p^2),\ D_1(P_3) = Gp^2.$$

As we have $D_1(P_l) = -D_2(P_l)$ for $l = 0,1,2,3$, the first-order conditions (13), (9) we can rewrite as

$$D_1(U_1) = -u_1^1G(1-4p+3p^2)+u_2^1G(2p-3p^2)+Gp^2 = 0 \tag{25}$$

$$D_2(U_2) = u_1^2G(2p-3p^2)-u_2^2G(1-4p+3p^2)+G(2p-p^2-1) = 0. \tag{26}$$

As $u_1^i < u_2^i + 1$, the only solution to both equations is at $G = 0$. Due to the satiating property, for some z there is $\bar{\alpha}$ and $\bar{\beta}$ such that for all $\|r\| = 1$, we have $r \cdot D_1(U_1) < 0$.

Since p_i are smooth functions, the vector field defined by $D_1(U_1(y,y))$ over X does not have singularities. Since all vectors $D_1(U_1(y,y))$ point inside the sphere with the center at z and of radius α, there must be some y^* inside the sphere at which $D_1(U_1(y^*,y^*)) = 0$.

Proof of Corollary 1.

We rewrite the condition (13) as

$$u_1'(y)\hat{p}_1'(0) + u_2'(y)\hat{p}_2'(0) + u_3'(y)\hat{p}_3'(0) = 0. \tag{27}$$

If $\hat{p}_1'(0) = \hat{p}_2'(0) = \hat{p}_3'(0) \neq 0$, then in any local maximum of U_U, the condition (27) is satisfied. If $u_i(\cdot)$ are concave, then the local maximum of U_U is unique, and is also a global maximum.

Proof of Corollary 2.

We rewrite the condition (13) as

$$u_1'(y)\hat{p}_1'(0)\hat{p}_2(0)\hat{p}_3(0) + u_2'(y)\hat{p}_1(0)\hat{p}_2'(0)\hat{p}_3(0) + u_3'(y)\hat{p}_1(0)\hat{p}_2(0)\hat{p}_3'(0) = 0 \tag{28}$$

or

$$u_1'(y)\frac{\hat{p}_1'(0)}{\hat{p}_1'(0)} + u_2'(y)\frac{\hat{p}_2'(0)}{\hat{p}_2'(0)} + u_3'(y)\frac{\hat{p}_3'(0)}{\hat{p}_3'(0)} = 0. \tag{29}$$

If $\hat{p}_1'(0) = \hat{p}_2'(0) = \hat{p}_3'(0) > 0$ and $\hat{p}_1(0) = \hat{p}_2(0) = \hat{p}_3(0) > 0$, then in any local maximum of U_N, the condition (29) is satisfied. If $u_i(\cdot)$ are concave, then the local maximum of U_N is unique, and is also a global maximum.

Proof of Proposition 2.

Transforming (9), we get

$$D_2(U_2) = (1 - u_2^2)D_1(P_1) + (1 - u_1^2)D_1(P_2) + D_1(P_3) = 0. \tag{30}$$

This equation is equivalent to (13) when (16) hold. It remains to be shown that (13) is satisfied somewhere. Fix $z \in \mathbf{R}^{\mathbf{k}}$. Let there be $\alpha > 0$ such that for all $r \in \mathbf{R}^2$, $|r| = 1$, we have (17): $r\dot{D}_1(U_1) < 0$. Since p_i are smooth functions, the vector field defined by $D_1(U_1)$ over X does not have singularities. Since all vectors $D_1(U_1(y,y))$ point inside the sphere with the center at z and of radius α, there must be some y^* inside the sphere at which $D_1(U_1(y^*,y^*)) = 0$.

If the condition fails (17) fails, then for all $z \in X$ there exists $\|r\| = 1$ such that $r\dot{D}_1(U_1(z + \alpha r, z + \alpha r)) > 0$. Obviously no z can be a local equilibrium.

Proof of Corollary 3.

Let $u_1^2 = 1 - u_2^2$, $u_1^2 = 1 - u_2^1$. Then we have $u_1^1 = 1 - u_2^1$, and

$$\begin{aligned} & u_1^1(1 - 2p_2 - 2p_3 + 3p_2p_3) + u_2^1(p_2 + p_3 - 3p_2p_3) + p_2p_3 = \\ = \; & u_1 + (1 - 3u_1)(p_2 + p_3 - 2p_2p_3) > 0, \end{aligned} \tag{31}$$

since $u_1 \leq \frac{1}{2}$ and $0 < p_2 + p_3 - 2p_2p_3 < 1$.

References

Banks, Jeffrey, and John Duggan. 2005. "Probabilistic Voting in the Spatial Model of Elections: The Theory of Office-Motivated Candidates." in David Austen-Smith and John Duggan, eds., *Social Choice and Strategic Decisions.* Springer, New York, NY

Coughlin, Peter and Shmuel Nitzan. 1981. "Electoral Outcomes with Probabilistic Voting and Nash Social Welfare Maxima." *Journal of Public Economics* 15: 113–121

Downs, Anthony. 1957. *An Economic Theory of Democracy*, New York: Harper and Row.

Duggan, John. 2000. "Equilibrium Equivalence Under Expected Plurality and Probability of Winning Maximization," mimeo, University of Rochester.

Groseclose, Timothy. 2001. "A model of candidate location when one candidate has a valence advantage." *American Journal of Political Science* 45(5): 862–886

Hinich, Melvin. 1977. "Equilibrium in spatial voting: The median voter result is an artifact." *Journal of Economic Theory* 16: 208–219

Hinich, Melvin, Ledyard, John, and Ordeshook, Peter. 1972. "Nonvoting and the existence of equilibrium under majority rule." *Journal of Economic Theory* 4: 144–153

Ledyard, John. 1984. The Pure Theory of Large Two-Candidate Elections. *Public Choice* 44: 7–41

Lin, Tse-Min, James Enelow, and Han Dorussen. 1999. "Equilibrium in Multicandidate Probabilistic Spatial Voting." *Public Choice* 98: 59–82

MasCollel, Andreu, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic*

Theory. Oxford University Press.

McKelvey, Richard, and John W. Patty. 2006. "A Theory of Voting in Large Elections." *Games and Economic Behavior*, 57(1): 155–180

Patty, John W. 2005. "Local Equilibrium Equivalence in Probabilistic Voting Models" *Games and Economic Behavior* 51(1): 523–536

Patty, John W. 2007. "Generic Difference of Expected Vote Share and Probability of Victory Maximization in Simple Plurality Elections with Probabilistic Voters." *Social Choice and Welfare*, 29(1): 149–173

Patty, John W, James M. Snyder, and Michael M. Ting. 2008. "Two's a Company, Three's An Equilibrium: Strategic Voting and Multicandidate Elections." Mimeo, Harvard University.

Schofield, Norman. 2007. "The Mean Voter Theorem: Necessary and Sufficient Conditions for Convergent Equilibrium." *Review Of Economic Studies* 74: 965–980

Schofield, Norman, and Itai Sened. 2006. *Multiparty Democracy: Elections and Legislative Politics,* Cambridge University Press.

Zakharov, Alexei V. 2008. "Candidate Location and Endogenous Valence." *Public Choice*, forthcoming